A characterization of the unique path lifting property for the whisker topology

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Introduction

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- Coverings for uniform spaces and/or topological groups [Lubkin,62][Berestovskii, Plaut,01,07,11][Brodskiy, Dyda, Labuz, Mitra,10]
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- Coverings defined in terms of unique lifting properties [Fischer, Zastrow,07][Brodskiy, Dyda, Labuz, Mitra,12][Dydak,11]

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Disk-coverings

Definition: [Dydak,11] A **disk-covering** is a map $p : E \to X$ such that for every $e \in E$

$$\begin{array}{ccc} & & (E, e) \\ & \overset{\exists\,!\,\widetilde{f_e}}{\nearrow} & \big\downarrow p \\ (D^2, x) & \xrightarrow[f]{} & (X, p(e)) \end{array}$$

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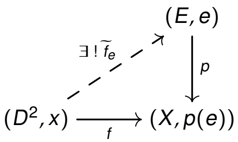

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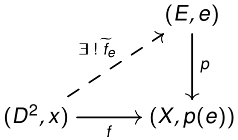
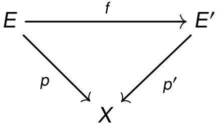


Since $[0, 1]$ is a retract of D^2, $p : E \to X$ uniquely lifts all paths just as a covering map does.

Let $\mathbf{DCov}(X)$ be the category of disk coverings. A morphism of disk-coverings over X is a commuting triangle



Disk-coverings

Every disk covering $p : E \to X$ yields a group action

$$\pi_1(X, x_0) \times p^{-1}(x_0) \to p^{-1}(x_0) \text{ given by } [\alpha] \cdot e = \widetilde{\alpha}_e(1)$$

with stabilizer $H = Im\,(p_* : \pi_1(E, e) \hookrightarrow \pi_1(X, x_0))$.

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Result: a faithful *monodromy* functor

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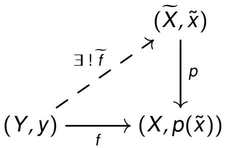
Typically, μ is not well-behaved (neither full nor essentially surjective).

Generalized covering maps

Definition: [Fischer, Zastrow,07] A **generalized covering map** is a map $p : \widetilde{X} \to X$ such that

1. $\widetilde{X}$ is path-connected and locally path-connected,
2. p is a continuous surjection,
3. For every path-connected locally path-connected space Y, point $\tilde{x} \in \widetilde{X}$, and map $f : (Y, y) \to (X, p(\tilde{x}))$ such that $f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(\widetilde{X}, \tilde{x}))$, there is a unique lift $\widetilde{f} : (Y, y) \to (\widetilde{X}, \tilde{x})$ such that $p \circ \widetilde{f} = f$.



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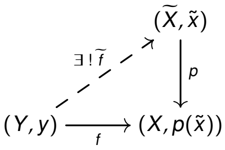

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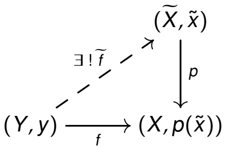
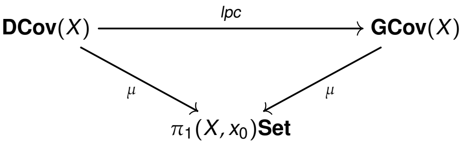


If $\widetilde{X}$ is simply connected, this is a generalized universal covering.
$\mathbf{GCov}(X) \subset \mathbf{DCov}(X)$ is the category of generalized coverings over X.

Generalized covering maps

Theorem: $\mu : \mathbf{GCov}(X) \to \pi_1(X, x_0)\mathbf{Set}$ is *fully* faithful and there is a coreflection functor $lpc : \mathbf{DCov}(X) \to \mathbf{GCov}(X)$ such that the diagram of functors commutes up to natural isomorphism



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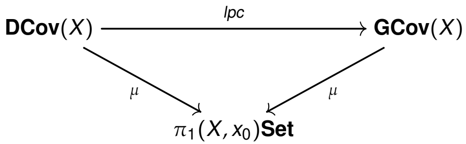

Consequences:

1. A generalized covering map $p : \widetilde{X} \to X$ is characterized *up to homeomorphism* by the stabilizer subgroup $H = p_*(\pi_1(\widetilde{X}, \tilde{x}))$.

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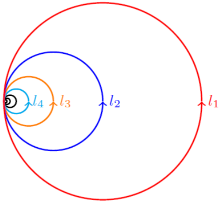
$$\begin{array}{ccc} \mathbf{DCov}(X) & \xrightarrow{\ lpc\ } & \mathbf{GCov}(X) \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & \pi_1(X, x_0)\mathbf{Set} & \end{array}$$

Consequences:

1. A generalized covering map $p : \widetilde{X} \to X$ is characterized *up to homeomorphism* by the stabilizer subgroup $H = p_*(\pi_1(\widetilde{X}, \tilde{x}))$.
2. *Any* attempt to characterize subgroups of $\pi_1(X, x_0)$ using maps which uniquely lift paths and homotopies of paths is absorbed by the theory of generalized coverings.

Existence of generalized covering subgroups

Example: [Fischer, Zastrow,07] Let $F_\infty = \langle [l_n] | n \geq 1 \rangle$ be the free group generated by loops traversing the n-th circle of the Hawaiian earring $\mathbb{H}$.



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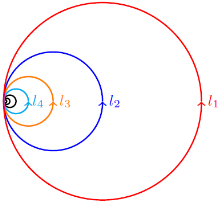

There cannot be a generalized covering $p : \widetilde{X} \to \mathbb{H}$ such that $p_*(\pi_1(\widetilde{X}, \tilde{x})) = F_\infty$ since any such map could not possibly have unique lifting:

$$l_1 \cdot l_2 \cdot l_3 \cdots \text{ and } \{l_1 \cdot l_2 \cdots l_n | n \geq 1\} \text{ lift to top. indistinguishable points in } p^{-1}(x_0)$$

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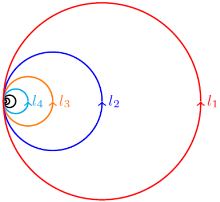

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Question: Which subgroups $H \leq \pi_1(X, x_0)$ correspond to generalized covering maps?

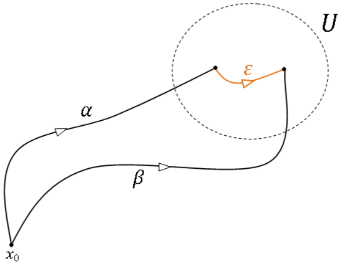
Identifying the topology of generalized coverings

The standard construction: Given $H \leq \pi_1(X, x_0)$, let $\widetilde{X}_H$ be the set of equivalence classes $[\alpha]_H$ of paths $\alpha : ([0,1], 0) \to (X, x_0)$.

$$\alpha \sim \beta \Leftrightarrow \alpha(1) = \beta(1) \text{ and } [\alpha \cdot \beta^{-}] \in H$$

Give $\widetilde{X}_H$ the **whisker topology** generated by basic sets

$$B([\alpha]_H, U) = \{[\alpha \cdot \epsilon]_H | \text{ where } Im(\epsilon) \subset U\} \text{ for open } U \subseteq X$$



Identifying the topology of generalized coverings

Definition: A map $f : X \to Y$ has the **unique path lifting property (UPL)** if whenever $f \circ \alpha = f \circ \beta$ for paths $\alpha, \beta : [0,1] \to X$ such that $\alpha(0) = \beta(0)$, then $\alpha = \beta$.

Lemma: [Fischer,Zastrow,07] If the endpoint projection $p_H : \widetilde{X}_H \to X$, $p_H([\alpha]_H) = \alpha(1)$ has UPL, then it is a generalized covering corresponding to H.

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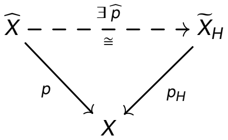
Lemma: If $p : \widehat{X} \to X$ is a generalized covering such that $H = p_*(\pi_1(\widehat{X}, \hat{x}))$, then



and thus $p_H : \widetilde{X}_H \to X$ has UPL.

Identifying the topology of generalized coverings

Theorem: For any subgroup $H \leq \pi_1(X, x_0)$, the following are equivalent:

1. There is a generalized covering $p : \widehat{X} \to X$ such that $H = p_*(\pi_1(\widehat{X}, \hat{x}))$,
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Certain cases have been confirmed [Brazas, Fabel,13][Fischer,Repovs,Virk,Zastrow,11][Fischer,Zastrow,07].

1. If $H = \pi^s(\mathscr{U}, x_0)$ is a Spanier group,
2. If $H = \ker(\pi_1(X, x_0) \to \check{\pi}_1(X, x_0))$,
3. If X is homotopy path-Hausdorff relative to H (i.e. closed in $\pi_1^{qtop}(X, x_0)$),
4. If $H = \bigcap_j H_j$ where each $p_j : \widetilde{X}_{H_j} \to X$ has UPL.

Main Result

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We consider a 1-dim. Peano continuum $\mathbb{D} \subset \mathbb{R}^2$,

a special free subgroup $\mathbb{F} \subset \pi_1(\mathbb{D}, d_0)$ and a *limit point* element $g_\infty \notin \mathbb{F}$,

and maps $f : (\mathbb{D}, d_0) \to (X, x_0)$ that separate $\mathbb{F}$ and g_∞.

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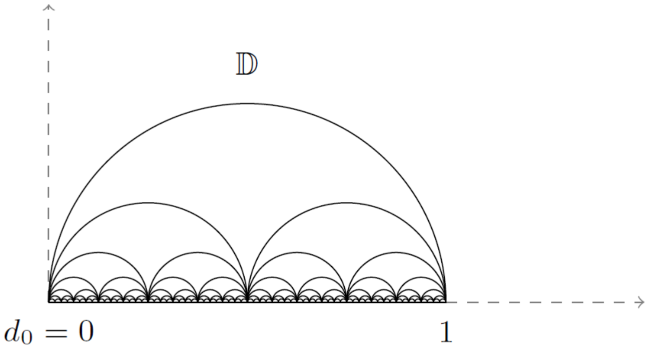
Theorem: For any subgroup $H \leq \pi_1(X, x_0)$, the following are equivalent:

1. $p_H : \widetilde{X}_H \to X$ has the unique path lifting property,
2. $f_*(\mathbb{F}) \subseteq H \Rightarrow f_*(g_\infty) \in H$ for every map $f : \mathbb{D} \to X$.

The space $\mathbb{D}$



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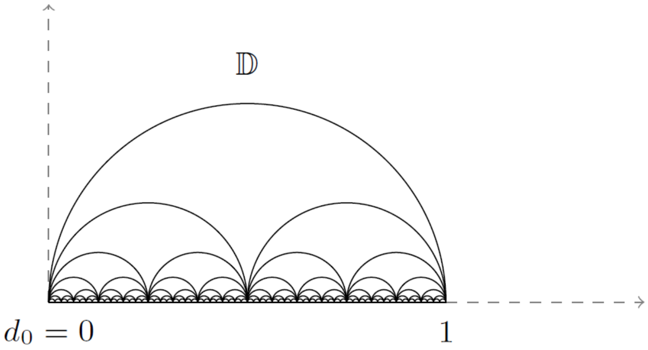

For each dyadic rational $\frac{2j-1}{2^n} \in (0,1)$, we add a semicircle of radius $\frac{1}{2^n}$.

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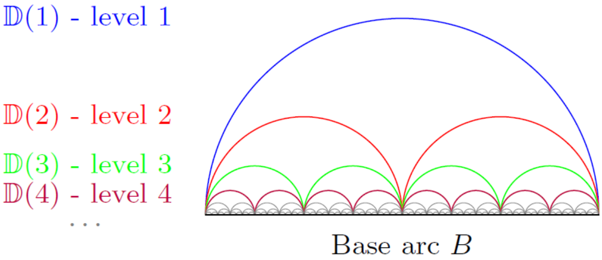

$$\mathbb{D} = B \cup \bigcup_{n \geq 1} \mathbb{D}(n)$$

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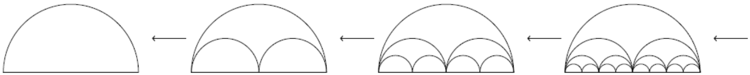


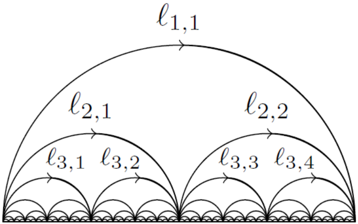
$$\mathbb{D} = \varprojlim_n \left[B \cup \bigcup_{k=1}^{n} \mathbb{D}(k) \right]$$

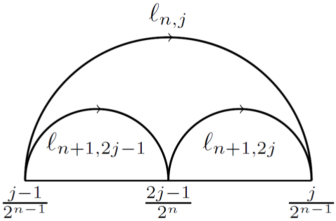
Since $\mathbb{D}$ is a one-dimensional, planar Peano continuum, we may inject $\pi_1(\mathbb{D}, d_0)$ into it's first shape group.

$$\pi_1(\mathbb{D}, d_0) \hookrightarrow \varprojlim_n \pi_1 \left(B \cup \bigcup_{k=1}^{n} \mathbb{D}(k), d_0 \right) = \varprojlim_n F_{2^{n+1}-1}$$

Paths in $\mathbb{D}$

Let $\ell_{n,j}$ be the path which traverses the j-th semicircle of the n-th level.

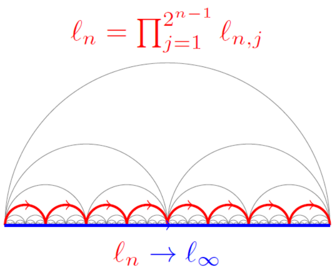


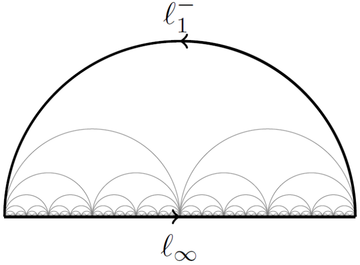


A **standard path** in $\mathbb{D}$ is a path of the form $\ell_{n,j}$ or its reverse $\left(\ell_{n,j}\right)^{-}$.

Paths in $\mathbb{D}$

Let ℓ_n be the path traversing the n-th level and ℓ_∞ be the unit speed path on the base.





$$g_\infty = \left[\ell_\infty \cdot \ell_1^-\right] \in \pi_1(\mathbb{D}, d_0).$$

Warm up: Homotopy path-Hausdorff

The homotopy path-Hausdorff property is stronger than the unique path lifting property.

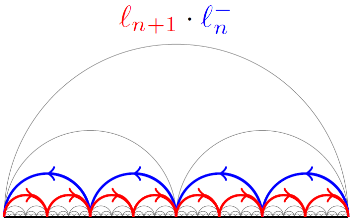
Definition: A locally path-connected metric space X is **homotopy path-Hausdorff relative to H** $\Leftrightarrow$ for every uniformly convergent sequence of loops $\alpha_n \to \alpha$ where $[\alpha_n] \in H$, $n \geq 1$, then $[\alpha] \in H$.

The subgroup $\mathbb{S} \leq \pi_1(\mathbb{D}, d_0)$

$\mathbb{S}$ is the free subgroup

$$\mathbb{S} = \langle [\ell_{n+1} \cdot \ell_n^-] \,|\, n \geq 1 \rangle.$$

Notice $g_\infty \notin \mathbb{S}$



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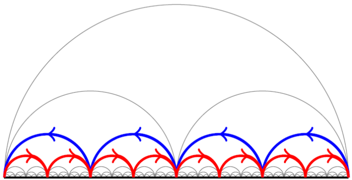

Theorem: For any subgroup $H \leq \pi_1(X, x_0)$, the following are equivalent:

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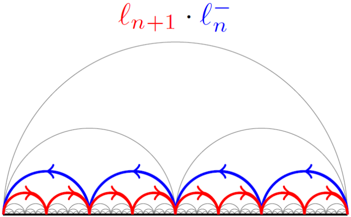

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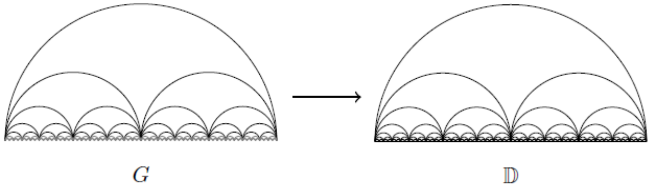
Idea of proof: $\ell_n \cdot \ell_1^- \to \ell_\infty \cdot \ell_1^-$ and $f_*([\ell_n \cdot \ell_1^-]) \in f_*(\mathbb{S}) \subseteq H$.

The subgroup $\mathbb{F} \leq \pi_1(\mathbb{D}, d_0)$

Consider $G = \bigcup_{n\geq 1} \mathbb{D}(n)$ with the CW-topology.



The subgroup $\mathbb{F}$ is the image of the inclusion

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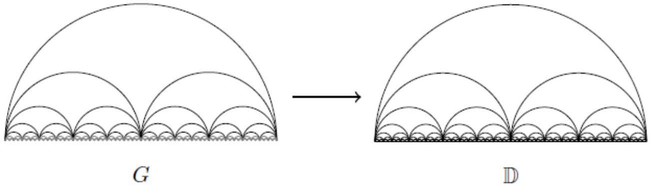

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$[\alpha] \in \mathbb{F} \Leftrightarrow \alpha$ is homotopic to a **finite** concatenation of standard paths.

The paths δ_t

Every real number $t \in [0,1]$ has a unique binary decimal expansion $t = 0.a_1 a_2 a_3....$, $a_n \in \{0,1\}$ which does not terminate in 1's.

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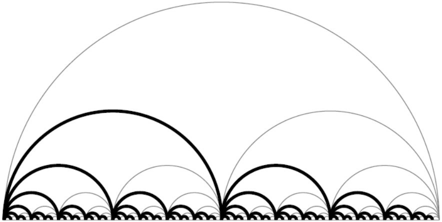
Construct a path δ_t in $\mathbb{D}$ from $d_0 = (0,0)$ to $(t,0)$ as follows:

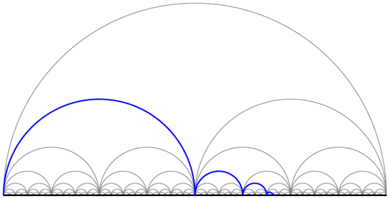


$$\text{Let } \delta_t = \prod_{n=1}^{\infty} \alpha_n \text{ where } \alpha_n = \begin{cases} \ell_{n+1,j_t} & \text{if } a_n = 1 \\ \text{constant} & \text{if } a_n = 0 \end{cases}$$

The paths δ_t

$$t = \frac{1}{\sqrt{2}} = 0.1011010100...$$

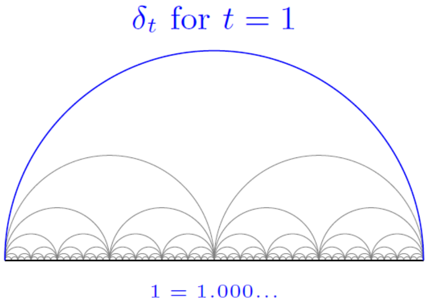
δ_t for $t = \frac{1}{\sqrt{2}}$



$$\frac{1}{2^1} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \cdots = \frac{1}{\sqrt{2}}$$

The paths δ_t

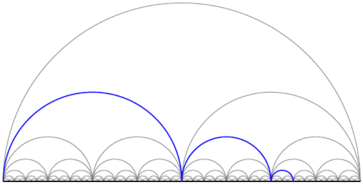
$t = 1$



The paths δ_t

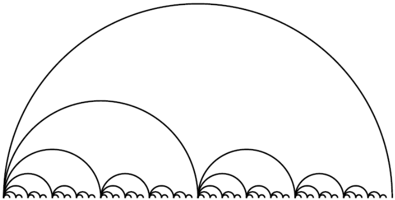
If $t \in (0,1)$, then δ_t is homotopic to a finite concatenation of standard paths $\Leftrightarrow$ t is a dyadic rational.

δ_t for $t = \frac{13}{16}$

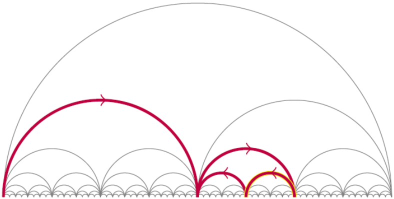


$\frac{13}{16} = 0.1101$

The subgroup $\mathbb{F} \leq \pi_1(\mathbb{D}, d_0)$

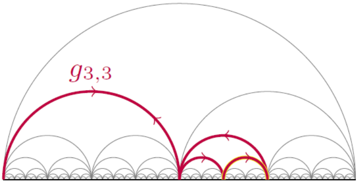


A maximal tree in G



Generators of $\pi_1(G, d_0)$

The subgroup $\mathbb{F} \leq \pi_1(\mathbb{D}, d_0)$

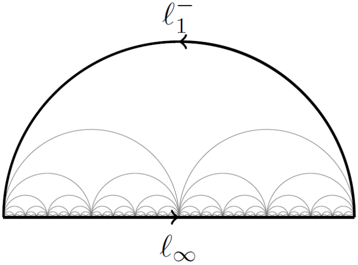


$$g_{n,j} = \left(\delta_{\frac{2j-1}{2^n}}\right) \cdot (\ell_{n+1,2j}) \cdot \left(\delta_{\frac{2j}{2^n}}\right)^{-}$$

$$\mathbb{F} = \langle g_{n,j} \,|\, n \geq 1,\ 1 \leq j \leq 2^{n-1} \rangle$$

The element g_∞

$$g_\infty = \left[\ell_\infty \cdot (\ell_1)^-\right]$$



Notice that $g_\infty \notin \mathbb{F}$

Idea of the proof

Theorem: For any subgroup $H \leq \pi_1(X, x_0)$, the following are equivalent:

1. $p_H : \widetilde{X}_H \to X$ has the unique path lifting property,
2. $f_*(\mathbb{F}) \subseteq H \Rightarrow f_*(g_\infty) \in H$ for every map $f : \mathbb{D} \to X$.

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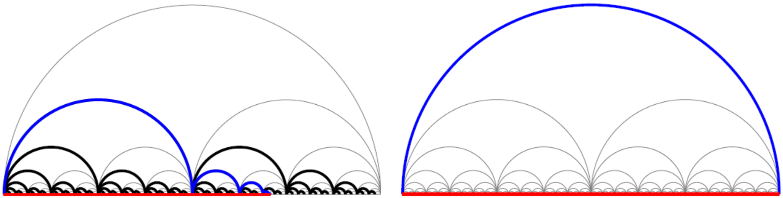
Idea of proof. (1) $\Rightarrow$ (2) Given $f : \mathbb{D} \to X$ with $f_*(\mathbb{F}) \subseteq H$ and $f_*(g_\infty) \notin H$, consider the path $\alpha(t) = f(t, 0)$ along the base arc.

$$\widetilde{\alpha}(t) = \left[\alpha|_{[0,t]}\right]_H \text{ and } \beta(t) = [f \circ \delta_t]_H$$

are distinct, continuous lifts $[0, 1] \to \widetilde{X}_H$ of α.



Idea of the proof

(2) $\Rightarrow$ (1) Suppose $\alpha : [0,1] \to X$ has a lift $\beta(t) = [\beta_t]_H$ different from $\widetilde{\alpha}(t) = \left[\alpha|_{[0,t]}\right]_H$ i.e. $[\beta_1]_H \neq [\alpha]_H$. Take a neighborhood of α in the compact-open topology:

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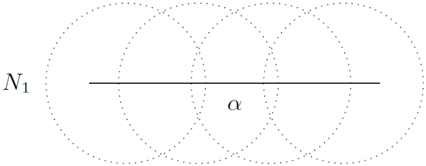

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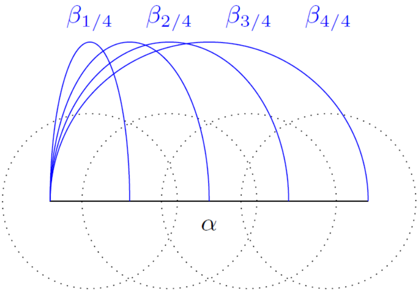

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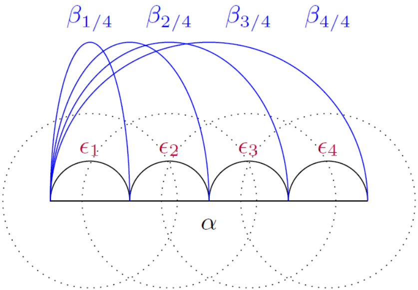

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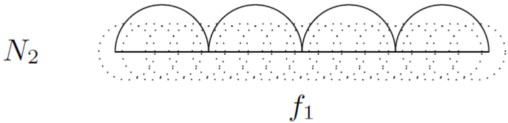

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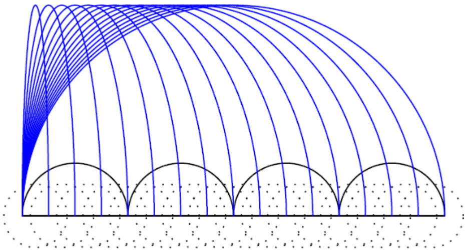

f_1

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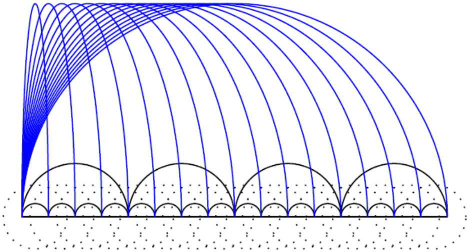

f_2

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(2) $\Rightarrow$ (1) Suppose $\alpha : [0,1] \to X$ has a lift $\beta(t) = [\beta_t]_H$ different from $\widetilde{\alpha}(t) = \left[\alpha|_{[0,t]}\right]_H$ i.e. $[\beta_1]_H \neq [\alpha]_H$. Continue inductively using a neighborhood base at α.

f

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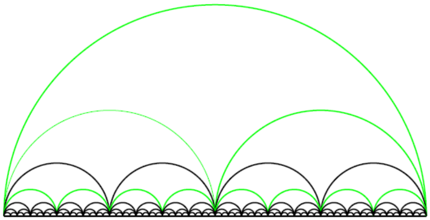

$$f : \mathbb{D} \to X$$

Generalized universal coverings

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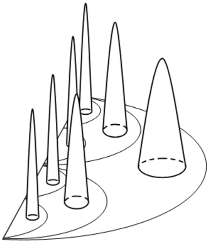
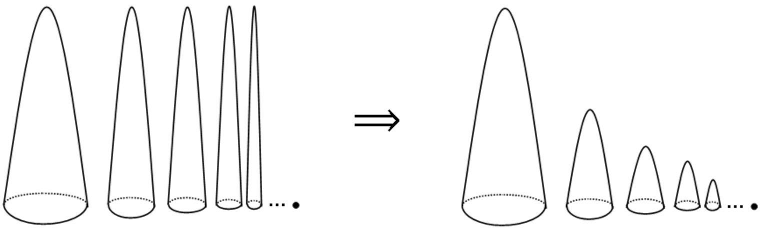
Corollary: For a path-connected metric space X, the following are equivalent:

1. X admits a generalized universal covering,
2. $p : \widetilde{X} \to X$ (standard construction) has UPL,
3. For every map $f : \mathbb{A} \to X$ from the archipelago-like space below, we must have $g_\infty \in \ker(f_*)$.



Sufficient conditions and examples

Corollary: Existence of a generalized universal covering is implied by the property (P): If $f_n : D^2 \to X$ is a sequence of maps where $f_n|_{S^1} \to z \in X$, then there exists $g_n : D^2 \to X$ such that $f_n|_{S^1} = g_n|_{S_1}$ and $g_n \to z$.



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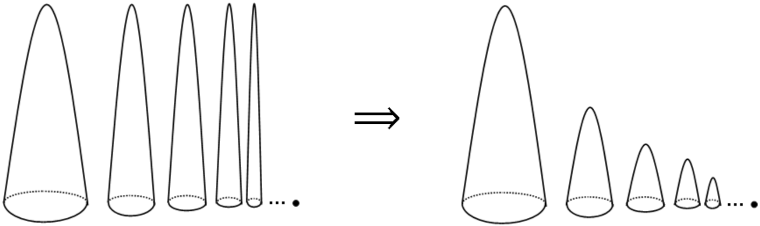

Property (P) $\Rightarrow$ $\exists$ generalized universal covering $\Rightarrow$ homotopy Hausdorff

Examples

1. The Sombrero space [Fischer,Repovs,Virk,Zastrow,11] and inverted sombrero [Zastrow, 11] spaces admit a generalized universal covering since they have property (P) [Conner, et. al., 08].

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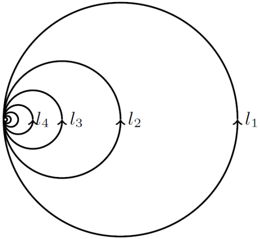
   Example: Take $X = \mathbb{D}$ and $H = \mathbb{S}$.

   $\mathbb{D}$ is not homotopy path-Hausdorff relative to $\mathbb{S}$,
   but $p_{\mathbb{S}} : \widetilde{\mathbb{D}}_{\mathbb{S}} \to \mathbb{D}$ has UPL since $\min\{diam(\alpha) | [\alpha] \in \mathbb{S}\} = 1$.

Necessary condition for $\mathbb{H}$



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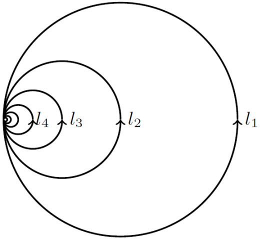

Let $F_\infty = \langle [\ell_n] | n \geq 1 \rangle \subseteq \pi_1(\mathbb{H}, x_0)$.

Corollary: If $K \leq \pi_1(\mathbb{H}, x_0)$ and $p_K : \widetilde{\mathbb{H}}_K \to \mathbb{H}$ has UPL, then H must satisfy the property: For every map $f : \mathbb{H} \to \mathbb{H}$ such that $f_*(F_\infty) \subseteq K$, then $f_*(\pi_1(\mathbb{H}, x_0)) \subseteq K$.

Thank you!